Reformulations of Quadratic Programs for Lipschitz Continuity

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***Abstract*— Optimization-based controllers often lack regularity guarantees, such as Lipschitz continuity, when multiple constraints are present. When used to control a dynamical system, these conditions are essential to ensure the existence and uniqueness of the system's trajectory. Here we propose a general method to convert a Quadratic Program (QP) into a Second-Order Cone Problem (SOCP), which is shown to be Lipschitz continuous. Key features of our approach are that (i) the regularity of the resulting formulation does not depend on the structural properties of the constraints, such as the linear independence of their gradients; and (ii) it admits a closed-form solution, which is not available for general QPs with multiple constraints, enabling faster computation. We support our method with rigorous analysis and examples. [Code][1]**

I. Introduction

This paper investigates the Lipschitz continuity of parametric Quadratic Programs (QPs) of the form

$$\min_{u\in\mathbb{R}^m} \ \|u-\pi_{\rm des}(x)\|^2 \tag{1a}$$
$$\text{s.t. } u\in K(x), \tag{1b}$$

where $x\in\mathcal{X}\subset\mathbb{R}^n$ is a parameter, and

$$K(x)=\{u\in\mathbb{R}^m : A(x)u\le b(x)\} \tag{2}$$

is a convex polyhedral set with $A:\mathcal{X}\to\mathbb{R}^{p\times m}$, $b:\mathcal{X}\to\mathbb{R}^p$, and $\pi_{\rm des}:\mathcal{X}\to\mathbb{R}^m$. We assume throughout that $K(x)$ is nonempty at each $x\in\mathcal{X}$. The set of minimizers of (1) is denoted by

$$\Pi(x)=\operatorname*{argmin}_{u\in K(x)}\|u-\pi_{\rm des}(x)\|^2, \tag{3}$$

where $\Pi:\mathcal{X}\rightrightarrows\mathbb{R}^m$ is a set-valued map. Since K is always nonempty and the objective function is strongly convex, $\Pi(x)$ contains exactly one element at each $x\in\mathcal{X}$. This unique minimizer is denoted by $\pi:\mathcal{X}\to\mathbb{R}^m$.

Lipschitz continuity of π, the (unique) minimizer of parametric problems such as (3), is of fundamental importance. For example, consider a nonlinear dynamical system

$$\dot{x}=f(x,u)=f(x,\pi(x))=f_{\rm cl}(x), \tag{4}$$

where the control input determined by (3), that is, $u=\pi(x)$. When π is Lipschitz continuous, its sensitivity to changes in x is bounded. That is, small perturbations in the state (induced by noise, numerical inaccuracies, etc.) result in bounded deviations in the control input, thereby preventing instability or chattering behavior of the system. Moreover, the Lipschitz continuity of π induces Lipschitz continuity of the closed-loop dynamics $f_{\rm cl}$. In turn, this ensures that the system (4) admits a unique solution for all time [1], guaranteeing that the system trajectories are unique.

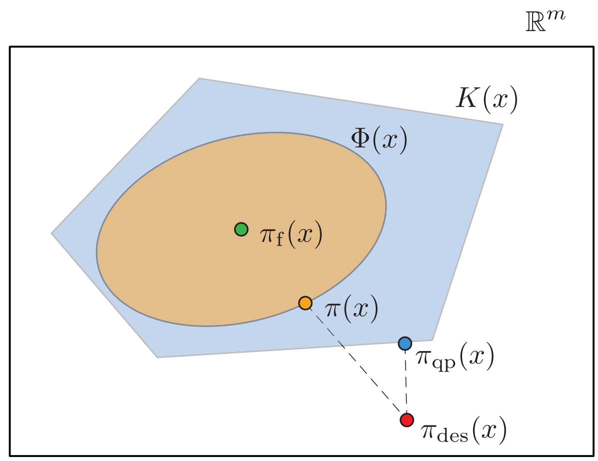

Fig. 1. A depiction of the main problem and our proposed solution approach. In parametric QPs, the solution $\pi_{\rm qp}(x)$ may be non-Lipschitz or even discontinuous with respect to x. This paper defines a subset $\Phi(x)$ of the constraint set $K(x)$ that is always feasible and Lipschitz continuous on $\mathbb{R}^m$ (defined in Assumption 2) such that the modified solution $\pi(x)$ is Lipschitz continuous. This is done by identifying a Lipschitz continuous function $\pi_{\rm f}(x)$ that is always feasible, and constructing a set $\Phi(x)$ around it such that $\pi_{\rm socp}(x)$ is Lipschitz wrt x.

However, it is well known that in the absence of certain constraint qualifications (e.g., Linear Independence Constraint Qualification (LICQ)), π can be non-Lipschitz or even discontinuous [2]–[4]. This is true even when $A,b,\pi_{\rm des}$ are continuously differentiable with respect to x (see Example 1), or Slater's Condition (SC) is satisfied (see Example 2).

Example 1. Let $x\in\mathcal{X}=\mathbb{R}$. Consider the QP

$$\pi_{\rm qp}(x)=\operatorname*{argmin}_{u\in\mathbb{R}^2}\left\|u-\begin{bmatrix}-2\\0\end{bmatrix}\right\|^2 \tag{5a}$$
$$\text{s.t. }\begin{bmatrix}1&0\\-1&-x\end{bmatrix}u\le\begin{bmatrix}1\\-(1+x)\end{bmatrix}. \tag{5b}$$

The solution of (5) can be derived analytically, and is plotted in Figure 2 (c). Even though A,b are continuously differentiable, notice that there exists a discontinuity of solution at $x=0$.

Since discontinuous solutions may not be preferable in practice, e.g., due to chattering phenomena, the focus of this

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[1]Code: https://github.com/joonlee16/Lipschitz-controllers

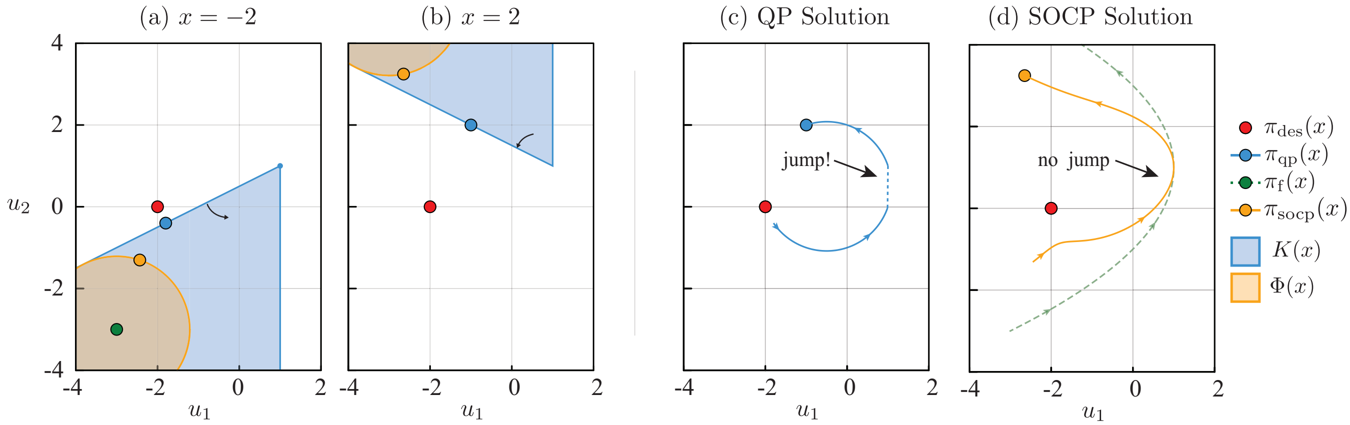

Fig. 2. Fig. (a) and (b) depict the feasible spaces and solutions of the original QP (5) and of our proposed SOCP reformulation in the form of (16) at $x=-2$ and $x=2$, respectively. Note at $x=0$, $K(x)$ shrinks to $\{u \in \mathbb{R}^2 : u_1 = 1\}$, a set with no interior. This causes π_{qp} to jump, as depicted in Fig. (c). In contrast, the SOCP reformulation introduces a smoothly varying feasible space $\Phi(x)$ centered at $\pi_{\text{f}}(x)$, ensuring that the solution $\pi_{\text{socp}}(x)$ transitions without jumps, as depicted in Fig. (d). Animations are available in our supplementary repository.

paper is to reformulate (1) into a problem of the form

$$\min_{u \in \mathbb{R}^m} \; \|u - \pi_{\text{des}}(x)\|^2 \tag{6a}$$

$$\text{s.t. } u \in \Phi(x) \subset K(x), \tag{6b}$$

whose minimizer is both unique and Lipschitz continuous with respect to x. To this end, we propose a suitable definition for $\Phi : \mathcal{X} \rightrightarrows \mathbb{R}^m$ and demonstrate that, by construction, its properties ensure the Lipschitz continuity of the minimizer of (6). Figure 1 depicts this pictorially.

Literature Review: Parametric optimization — both convex and nonconvex — has been extensively studied [5]–[8], with a central focus on establishing conditions under which the solution map exhibits regularity [2], [3], [9]–[11]. Under constraint qualifications such as LICQ, Mangasarian-Fromovitz Constraint Qualification (MFCQ), and SC, solutions to QPs can be differentiable, Lipschitz, or Hölder continuous, depending on the setting. See [8], [11] for detailed summary.

To apply these results, one must verify constraint qualifications hold. Although feasible in some cases [12]–[14], this is generally difficult. One alterative approach is to combine multiple constraints into a unified constraint [15]–[17], where LICQ holds trivially. However, this may render the problem infeasible even if $K(x)$ is non-empty.

Another line of work analyzes regularity of the feasible set map $K : \mathcal{X} \rightrightarrows \mathbb{R}^m$, treating it as a set-valued mapping [7], [18], [19]. This literature studies whether K is Lipschitz continuous and when Lipschitz selections of K exist.

Contributions: In this paper, we take a fundamentally different approach. Instead of focusing on the structural conditions of the QP like constraint qualifications, we

- propose a method to reformulate the QP (3) into a Second-Order Cone Problem (SOCP) (6) that is guaranteed to admit a unique Lipschitz continuous solution under mild assumptions,
- show that our SOCP admits a closed-form solution,
- illustrate the challenges and ideas through examples,
- provide a conjecture to expand the set of feasible solutions and reduce conservatism.

The core theoretical insight is that by carefully tightening the constraint set, we ensure Lipschitz continuity while preserving uniqueness. A key source of non-Lipschitz behavior in (3) is when two active constraints become linearly dependent, as shown in Example 1. Small perturbations in this case can drastically alter the feasible set and solution. Our formulation defines Φ to avoid such degeneracies.

II. PRELIMINARIES

Let $\mathbb{R}$ denote the set of reals. Let C^k denote the set of functions with k continuous derivatives.

Definition 1 ([8, Def 9.1])**.** *Let X, Y be normed spaces. A function $f : X \to Y$ is **Lipschitz continuous on** X if there exists a $L \geq 0$ such that*

$$\|f(x_1) - f(x_2)\| \leq L \|x_1 - x_2\|, \; \forall x_1, x_2 \in X. \tag{7}$$

*f is **locally Lipschitz at** $x \in X$ if there exists a neighborhood $\mathcal{B} \subset X$ of x and a $L = L(x) \geq 0$ such that*

$$\|f(x_1) - f(x_2)\| \leq L \|x_1 - x_2\|, \; \forall x_1, x_2 \in \mathcal{B}. \tag{8}$$

f is locally Lipschitz on $\mathcal{X}$ if it is locally Lipschitz $\forall x \in X$.

A Lipschitz function is also locally Lipschitz, while the converse is not necessarily true [20, Th. 2.1.6]. Lipschitz continuity can also be extended to set-valued maps.

Definition 2 ([7, Def. 9.26])**.** *A set-valued map $K : \mathcal{X} \rightrightarrows \mathbb{R}^m$ is **Lipschitz continuous on** $\mathcal{X} \subset \mathbb{R}^n$ if it is nonempty-closed-valued on $\mathcal{X}$ and there exists a $L \geq 0$ such that*

$$d_H(K(x_1), K(x_2)) \leq L \|x_1 - x_2\| \;\; \forall x_1, x_2 \in \mathcal{X}. \tag{9}$$

where d_H is the Hausdorff distance [7, Def. 4.13].

Parametric Convex Optimization

Recall problem (1). Let $\mathcal{J} = \{1, \dots, p\}$ denote the constraint indices. Let $a_i(x)$ and $b_i(x)$ denote the i^{th} row of $A(x)$ and $b(x)$ respectively, i.e., $a_i : \mathcal{X} \to \mathbb{R}^m$, $b_i : \mathcal{X} \to \mathbb{R}$.

The i^{th} constraint is active at x *iff* $b_i(x) = a_i(x)^\top \pi(x)$. A set of constraint with indices $\mathcal{I} \subset \mathcal{J}$ are linearly dependent at $x \in \mathcal{X}$ if vectors $a_i(x)$, $i \in \mathcal{I}$, are linearly dependent.

Various constraint qualifications have been studied in the literature, see [6], [7], [11]. Of particular interest is SC:

Definition 3 (Section 2.2 [11])**.** *For* (1)*,* ***Slater's Condition (SC)*** *holds at* $x \in \mathcal{X}$ *if there exists* $u \in \mathcal{U}$ *such that* $a_i(x)^\top u < b_i(x)$*, for all* $i \in \mathcal{J}$*.*

Although SC guarantees existence of a smooth selection [21, Proposition 3.1], finding this function remains an open problem. Under certain conditions, SC guarantees continuity (not local Lipschitz) of its solution mapping [9], [11], for example in scalar optimization problems [11, Proposition 3.3]. This does not extend to higher-dimensional cases. Lipschitz continuity of K (in the sense of Definition 2) is also not sufficient for solutions of (1) to be Lipschitz — solutions are only Hölder continuous [19], [22, Ex. 1.2].

The next example demonstrates a more subtle phenomenon, where even when the original problem satisfies strong regularity conditions such as SC, its minimizer may still fail to be Lipschitz.

Example 2. Consider the following problem inspired by [2]:

$$\underset{u \in \mathbb{R}^2}{\operatorname{argmin}} \ \|u\|^2 \tag{10a}$$

$$\text{s. t.} \begin{bmatrix} -1 & 0 \\ -1 & -x_1 \end{bmatrix} u \leq \begin{bmatrix} -1 \\ -(1 + x_2) \end{bmatrix} \tag{10b}$$

for $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathcal{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 2\}$. The problem admits a strictly feasible solution

$$\pi_{\text{f}}(x) = \begin{bmatrix} 2 + |x_2| & 0 \end{bmatrix}^\top, \tag{11}$$

and therefore SC is satisfied. However, for $x_2 \leq \frac{1}{2}x_1^2$, the solution is

$$\pi_{\text{qp}}(x) = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}^\top & \text{if } x_2 \leq 0, \\ \begin{bmatrix} 1 & \frac{x_2}{x_1} \end{bmatrix}^\top & \text{if } 0 < x_2 \leq \frac{1}{2}x_1^2, \end{cases} \tag{12}$$

which implies that (10) is not Lipschitz at $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$.

Problem Statement

Examples 1 and 2 illustrate that QP may not be (locally) Lipschitz continuous. These discontinuities stem from the unbounded sensitivity of the solutions about the point where their active constraints have linearly dependent rows in $A(x)$. Small perturbations in x can cause abrupt changes in the active constraint set, leading to sharp, potentially unbounded changes in the solution along the active affine constraints. Other similar examples can be found in [3], [11]. Thus, the goal of this paper is the following:

Problem 1. *Given an optimization problem* (1)*, design a* $\Phi : \mathcal{X} \rightrightarrows \mathbb{R}^m$ *such that* $\Phi(x) \subset K(x)$ *for all* $x \in \mathcal{X}$ *and the modified optimization problem* (6) *admits a unique solution* $\pi : \mathcal{X} \to \mathcal{U}$ *for every* $x \in \mathcal{X}$ *and such that* π *is Lipschitz.*

We make the following mild assumptions:

Assumption 1.

1) *Each row of* $A : \mathcal{X} \to \mathbb{R}^{p \times m}$ *has unit-norm, i.e.,* $\|a_i(x)\| = 1$, $\forall i \in \mathcal{J}$*.*
2) *The functions* $a_i : \mathcal{X} \to \mathbb{R}^m$*,* $b_i : \mathcal{X} \to \mathbb{R}$*, and* $\pi_{\text{des}} : \mathcal{X} \to \mathbb{R}^m$ *are Lipschitz continuous, with constants* $L_{a_i}, L_{b_i}, L_{\pi_{\text{des}}} \geq 0$ *respectively.*

III. SOLUTION

A. Main Result

Our key idea is that if we know that there exists a $\pi_{\text{f}} : \mathcal{X} \to \mathcal{U}$ that is Lipschitz on $\mathcal{X}$ and satisfies the constraints, we can exploit this solution to construct a solution to Problem 1. We formalize this as follows:

Assumption 2. *There exists* $\pi_{\text{f}} : \mathcal{X} \to \mathcal{U}$ *such that*

$$\pi_{\text{f}}(x) \in K(x) \quad \forall x \in \mathcal{X}, \tag{13}$$

π_{f} *is Lipschitz on* $\mathcal{X}$*, and* π_{f} *is bounded:* $\exists \overline{U}_{\text{f}} > 0$ *such that* $\|\pi_{\text{f}}(x)\| \leq \overline{U}_{\text{f}}$ $\forall x \in \mathcal{X}$*.*

Remark 1. Note that π_{f} directly addresses Problem 1: $\Phi(x) = \{\pi_{\text{f}}(x)\}$ is a valid solution to Problem 1. However, π_{f} does not consider the objective of the original QP and thus can lie far from the desired solution. As shown in Figure 1, the new formulation (16) leverages π_{f} to reduce this conservatism, while preserving Lipschitz continuity. While such a π_{f} might not be easily available for general problems, we propose several methods to construct π_{f} and thus relax the assumption later in Section III-B.

We propose the following definition for Φ:

$$\Phi(x) = \{u \in \mathbb{R}^m : g_i(x, u) \leq 0, \forall i \in \mathcal{J}\}, \tag{14}$$

where

$$g_i(x, u) = \|u - \pi_{\text{f}}(x)\| - (b_i - a_i(x)^\top \pi_{\text{f}}(x)). \tag{15}$$

This corresponds to the following modified problem:

$$\pi_{\text{socp}}(x) = \underset{u \in \mathbb{R}^m}{\operatorname{argmin}} \ \|u - \pi_{\text{des}}(x)\|^2 \tag{16a}$$

$$\text{s. t.} \ \|u - \pi_{\text{f}}(x)\| \leq r(x) \tag{16b}$$

where

$$r(x) = \min_{i \in \mathcal{J}} \ (b_i(x) - a_i(x)^\top \pi_{\text{f}}(x)). \tag{17}$$

The $K(x)$ and $\Phi(x)$ sets are depicted in Figure 2. The blue region shows $K(x)$, while the orange region represents $\Phi(x)$ of the proposed reformulation. $\Phi(x)$ is a ball centered at $\pi_{\text{f}}(x)$ with radius $r(x)$.

While the original problem (1) is a QP, (16) is a SOCP. Both are convex problems that can be solved efficiently [23]. However, in this case, we can also show that SOCP has a closed-form solution. Using a shift of variables, the solution to (16) can equivalently be expressed as

$$\pi_{\text{socp}}(x) = \pi_{\text{f}}(x) + v(x), \tag{18}$$

where $v : \mathcal{X} \to \mathbb{R}^m$ is given by

$$v(x) = \underset{v \in \mathbb{R}^m}{\operatorname{argmin}} \|v - v_{\text{des}}(x)\|^2 \;\; \text{s.t.} \;\; \|v\| \leq r(x),$$

and $v_{\text{des}}(x) = \pi_{\text{des}}(x) - \pi_{\text{f}}(x)$. This leads to the following closed-form solution for π_{socp}:

$$\pi_{\text{socp}}(x) = \pi_{\text{f}}(x) + \min\left(r(x), \|v_{\text{des}}(x)\|\right) \frac{v_{\text{des}}(x)}{\|v_{\text{des}}(x)\|}. \quad (19)$$

We characterize (16) as our main result:

Theorem 1. *Let Assumption 1, 2 hold. Then,*

1) *there exists a unique minimizer of the optimization problem (16) for all $x \in \mathcal{X}$. Let $\pi_{\text{socp}} : \mathcal{X} \to \mathcal{U}$ denote this solution.*
2) *The solution satisfies $\pi_{\text{socp}}(x) \in K(x)$ for all $x \in \mathcal{X}$.*
3) *The solution π_{socp} is Lipschitz on $\mathcal{X}$.*

Proof. [First Claim] By Assumption 2, the feasible space for (16) is non-empty: $\pi_{\text{f}}(x) \in \Phi(x)$. Since the objective $\|u - \pi_{\text{des}}(x)\|^2$ is strongly convex in u, there must exist a unique minimizer [23, Section 4.2.1].

[Second Claim] Consider any feasible solution $\pi \in \Phi(x)$. By Assumption 1.1, we know $\|a_i(x)\| = 1$ for $i \in \mathcal{J}$. Also, since the solution satisfies (16) for all $i \in \mathcal{J}$,

$$\begin{aligned} b_i(x) - a_i(x)^\top \pi_{\text{f}}(x) &\geq \|\pi - \pi_{\text{f}}(x)\| \\ &= \|a_i(x)\| \, \|\pi - \pi_{\text{f}}(x)\| \\ &\geq a_i(x)^\top (\pi - \pi_{\text{f}}(x)) \end{aligned}$$

Therefore $b_i(x) \geq a_i(x)\pi$, i.e., $\pi \in K(x)$.

[Third Claim] Notice v_{des} is Lipschitz with constant $L_{v_{\text{des}}} = L_{\pi_{\text{des}}} + L_{\pi_{\text{f}}}$. Recall $r(x) = \min_{i \in \mathcal{J}}(r_i(x))$, where $r_i(x) = b_i(x) - a_i(x)^\top \pi_{\text{f}}(x)$. Therefore, each r_i has Lipschitz constant[2] $L_{r_i} = L_{b_i} + (L_{\pi_{\text{f}}} + L_{a_i}\overline{U_f})$. Therefore by [20, Prop. 2.3.9], r has the Lipschitz constant $L_r = \max_{i \in \mathcal{J}}(L_{r_i})$. Finally, using Lemma 2 we know that v has Lipschitz constant $L_v = L_{v_{\text{des}}} + L_r$. Hence π_{socp} is Lipschitz with constant $L = L_{\pi_{\text{f}}} + L_v = L_{\pi_{\text{des}}} + 2L_{\pi_{\text{f}}} + L_r$. □

The proof relies on Lemma 2, provided in the appendix.

Remark 2. Note that Theorem 1 does not rely on the structural properties or conditions of the QP, such as constraint qualifications. Instead, we reformulate it as an SOCP in the form of (16), which by construction is guaranteed to yield a Lipschitz-continuous solution.

We can also establish local Lipschitz continuity:

Corollary 1. *Suppose Assumption 1 holds except that $a_i, b_i, \pi_{\text{des}}$ are locally Lipschitz, and that $\pi_{\text{f}} : \mathcal{X} \to \mathcal{U}$ is locally Lipschitz and satisfies $\pi_{\text{f}}(x) \in K(x)$ for all $x \in \mathcal{X}$. Then Claims 1 and 2 of Theorem 1 hold, and π_{socp} is locally Lipschitz continuous on $\mathcal{X}$.*

The proof is omitted for brevity but follows that of Theorem 1. Notice here u_f need not be bounded.[3]

B. Constructing the feasible solution π_{f}

Here we relax Assumption 2 by providing methods to construct π_{f}. Naturally, the performance of our SOCP (16) depends on π_{f}: the farther π_{f} lies from the boundary of the feasible set $K(x)$, the larger $\Phi(x) \subset K(x)$, and consequently, the closer $\pi_{\text{socp}}(x)$ can be to the desired $\pi_{\text{des}}(x)$. Although π_{f} can be hand-crafted for specific problems (as in the examples above), we identify two candidates for π_{f} for general problems: the Analytic center and the Steiner point.

Analytic Center: Suppose $K(x)$ as defined in (2) is compact and has nonempty interior. Then the analytic center [23, Ex. 4.6] of $K(x)$ is the unique solution to

$$\operatorname{Ac}(K(x)) = \underset{u \in \mathbb{R}^m}{\operatorname{argmin}} \; -\sum_{i=1}^{p} \log\left(b_i(x) - a_i(x)u\right). \quad (20)$$

This can be used to define the feasible solution π_{f}:

Corollary 2. *Suppose (i) Assumption 1 holds, (ii) $a_i, b_i \in C^2$, (iii) K is compact-valued, and (iv) SC holds for (1) for all $x \in \mathcal{X}$. Then, if $\pi_{\text{f}}(x) = \operatorname{Ac}(K(x))$, π_{socp} in (16) is locally Lipschitz continuous on $\mathcal{X}$.*

Proof. By Lemma 1 (see Appendix), π_{f} is locally Lipschitz on $\mathcal{X}$. Hence the claim holds by Corollary 1. □

Steiner Point: The Steiner point is a Lipschitz selection of K provided the set-valued map K is Lipschitz in the sense of Definition 2 [18, Section 2.4.3]. The Steiner point of a nonempty compact convex set $C \subset \mathbb{R}^m$ is defined by

$$\operatorname{St}(C) = \frac{1}{\int_B d\mu} \int_B \nabla \sigma_C \, d\mu, \quad (21)$$

where B is the unit ball in $\mathbb{R}^m$, μ is the Lebesgue measure, and $\sigma_C : \mathbb{R}^m \to \mathbb{R}$ is the support function of C [18, Eq. 2.16]. For $K : \mathcal{X} \rightrightarrows \mathbb{R}^m$ with Lipschitz constant L_K, the Lipschitz constant for $x \mapsto \operatorname{St}(K(x))$ is $L = mL_K$.

Corollary 3. *Suppose $K : \mathcal{X} \rightrightarrows \mathbb{R}^m$ is Lipschitz continuous and compact for all $x \in \mathcal{X}$. If $\pi_{\text{f}}(x) = \operatorname{St}(K(x))$, then π_{socp} in (16) is Lipschitz continuous on $\mathcal{X}$.*

Note, even when K is Lipschitz, it is known that π_{qp} as in (1) is only Hölder continuous (see [19], [22, Ex. 1.2]). Our proposed SOCP yields Lipschitz continuous solutions.

C. A Conjecture

We now propose a convex Quadratically Constrained Quadratic Program (QCQP) that we conjecture is also Lipschitz continuous. Consider the following definition for Φ:

$$\Phi_{\text{qcqp}}(x) = \{u \in \mathbb{R}^m : h_i(x, u) \leq 0\} \quad (22)$$

where for any fixed $k > 0$,

$$h_i(x, u) = \|u - \pi_{\text{f}}(x)\|^2 - 2k\left(b_i(x) - a_i(x)^\top u\right). \quad (23)$$

[2]Recall that for $f, g : X \to Y$, the Lipschtiz constant of fg is $L = (\sup_{x \in X} \|g(x)\|)L_f + (\sup_{x \in X} \|f(x)\|)L_g$. See [20, Prop. 2.3.4].

[3]In particular, the boundedness of π_{f} is only needed to ensure r_i is Lipschitz. When a_i and π_{f} are both locally Lipschitz, r_i is also locally Lipschitz, even when π_{f} is unbounded [20, Prop. 2.3.4].

Compare this with g_i in (15). There are three key differences: (i) the norm is squared in the first term, (ii) we introduced a tuning parameter $k > 0$, (iii) the last term is $a_i(x)^\top u$, not $a_i(x)^\top \pi_{\mathrm{f}}(x)$. This changes the structure of $\Phi(x)$ from an intersection of concentric balls centered at $\pi_{\mathrm{f}}(x)$ to an intersection of non-concentric balls containing the point $\pi_{\mathrm{f}}(x)$. This design expands the feasible space and introduces a tuning parameter that allows the user to control the conservatism. By some algebra, the modified problem is

$$\underset{u\in\mathbb{R}^m}{\operatorname{argmin}} \ \|u - \pi_{\mathrm{des}}(x)\|^2 \tag{24a}$$
$$\text{s.t.} \ \|u - c_i(x)\|^2 \le d_i(x), \tag{24b}$$

where

$$c_i(x) = \pi_{\mathrm{f}}(x) - k a_i(x),$$
$$d_i(x) = k^2 + 2k\left(b_i(x) - a_i(x)^\top \pi_{\mathrm{f}}(x)\right).$$

Conjecture 1. *Let Assumption 1, 2 hold and $k > 0$. Then,*

1) *there exists a unique minimizer of the optimization problem (24) for all $x \in \mathcal{X}$. Let $\pi_{\mathrm{qcqp}} : \mathcal{X} \to \mathcal{U}$ denote this solution.*
2) *The solution satisfies $\pi_{\mathrm{qcqp}}(x) \in K(x)$ for all $x \in \mathcal{X}$.*
3) *The solution π_{qcqp} is Lipschitz on $\mathcal{X}$.*

Partial Proof. [First Claim] Notice that since $b_i(x) - a_i(x)^\top \pi_{\mathrm{f}}(x) \ge 0$ by Assumption 2 and $k > 0$, we have $d_i(x) \ge k^2$. Since $\|a_i(x)\| = 1$ by Assumption 1.1, $u = \pi_{\mathrm{f}}(x)$ is a feasible solution. Since the objective is strictly convex, there always exists a unique solution to (24).

[Second Claim] Consider any feasible solution u. Then $h_i(x, u) \le 0$. Then, $\|u - \pi_{\mathrm{f}}(x)\|^2 \le 2k(b_i(x) - a_i(x)^\top u)$, which can only be satisfied if $b_i(x) - a_i(x)^\top u \ge 0$. Since this is true for all $i \in \mathcal{J}$, $u \in K(x)$.

[Third Claim] Proving this remains an open question. □

Importantly, one can construct an example where if instead the constraints $\tilde{h}_i(x, u) = \|u - \pi_{\mathrm{f}}(x)\| - 2k(b_i(x) - a_i(x)^\top u)$ are used, the solution can be discontinuous. The square on the norm in (23) appears to be important.

IV. Case Studies

In this section, we demonstrate our approach using our earlier examples. Recall Example 1. For $x \in \mathcal{X} = [-2, 2]$, the analytic solution to the QP is given by

$$\pi_{\mathrm{qp}}(x) = \begin{cases} \begin{bmatrix} 1 & 1 \end{bmatrix}^\top & \text{if } x \in (0, 1/3], \\ \begin{bmatrix} \frac{1+x-2x^2}{1+x^2} & \frac{3x+x^2}{1+x^2} \end{bmatrix}^\top & \text{else} \end{cases} \tag{25}$$

which is discontinuous at $x = 0$, as depicted in Figure 2. Notice that this problem admits a Lipschitz solution

$$\pi_{\mathrm{f}}(x) = \begin{bmatrix} 1 - x^2 & 1 + 2x \end{bmatrix}^\top. \tag{26}$$

Using this feasible solution, we can formulate a SOCP using (16) that yields a Lipschitz continuous solution, depicted in Figure 2 (d)).

Similarly, we can reformulate Example 2 into SOCPs using (16). The analytic solution for Example 2 is

$$\pi_{\mathrm{socp}}(x) = \begin{bmatrix} 2 + |x_2| - \min\left(1 + |x_2|, \frac{1 - x_2 + |x_2|}{\sqrt{1 + x_1^2}}\right) \\ 0 \end{bmatrix}$$

which is Lipschitz on $\mathcal{X}$. A similar but more complicated expression is available for Example 1 in the code repository.[4]

Now, we consider Robinson's famous counterexample [2]:

$$\underset{u\in\mathbb{R}^4}{\operatorname{argmin}} \ \frac{1}{2}\|u\|^2 \tag{27a}$$
$$\text{s.t.} \ \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & -x_1 \end{bmatrix} u \le \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 - x_2 \end{bmatrix} \tag{27b}$$

for $x \in \mathcal{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \le 2\}$. Although (27) is well-posed (satisfying SC, strictly convex objective, etc), it was proven in [2] that the solution is not locally Lipschitz continuous at $x = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$. It has a feasible solution $\pi_{\mathrm{f}}(x) = \begin{bmatrix} 0 & 0 & 2 + |x_2| & 0 \end{bmatrix}^\top$, that is Lipschitz continuous and bounded on $\mathcal{X}$. We can use the π_{f} above to reformulate (27) into an SOCP form, and obtain a π_{socp} that is Lipschitz continuous by Theorem 1. Alternatively, we may construct the SOCP using the analytic center (20) which guarantees local Lipschitz continuity by Corollary 2. Before transforming (27) into the SOCP with the analytic center, note that its feasible space $K(x) = \{u \in \mathbb{R}^4 : A(x)u \le b(x)\}$ is not compact, violating one of the assumptions in Corollary 2. To ensure the compactness, we introduce bounding box constraints $\|u\|_\infty \le u_{\max}$, with $u_{\max} = 5$ chosen large enough so that they are always inactive. We can now use the analytic center and solve (16). The π_{socp} is locally Lipschitz continuous on $\mathcal{X}$.

V. Conclusions

In this letter, we propose a new framework for optimization-based control synthesis that ensures a locally Lipschitz continuous control law. To this end, we reformulate a given Quadratic Program (QP), a common form of optimization-based controllers, into a Second-Order Cone Problem (SOCP). This reformulation guarantees Lipschitz continuity without requiring specific structural assumptions on the constraint matrices and admits an analytical solution form. As future work, we aim to develop an extended framework that further relaxes assumptions and verifies the conjecture. We also plan on using these results in multi-agent optimization-based controllers.

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APPENDIX

Lemma 1. *Let all assumptions in Corollary 2 hold. Then, the function $\pi_{\mathrm{f}} : \mathcal{X} \to \mathbb{R}^m$ defined by $\pi_{\mathrm{f}}(x) = \mathrm{Ac}(K(x))$ is locally Lipschitz on $\mathcal{X}$, and $\pi_{\mathrm{f}}(x) \in K(x)$.*

Proof. Let $G(u, x) = -\sum_{i=1}^{p} \log(b_i(x) - a_i(x)u)$. By SC, $K(x)$ has a non-empty interior at each $x \in \mathcal{X}$. Hence, for $\bar{x} \in \mathcal{X}$, $\bar{u} = \pi_{\mathrm{f}}(\bar{x})$ is well-defined and $\bar{u} \in K(\bar{x})$. Since $\log(\cdot)$ is smooth,

$$\nabla_u G(\bar{u}, \bar{x}) = \sum_{i=1}^{p} \frac{a_i(\bar{x})}{b_i(\bar{x}) - a_i(\bar{x})^\top \bar{u}} = 0,$$

$$\nabla_{uu} G(\bar{u}, \bar{x}) = \sum_{i=1}^{p} \frac{a_i(\bar{x})^\top a_i(\bar{x})}{(b_i(\bar{x}) - a_i(\bar{x})^\top \bar{u})^2} > 0.$$

Both expressions are well defined since by SC, $\bar{u}$ satisfies $a_i(\bar{x})^\top \bar{u} < b_i(\bar{x})$. Similar expressions can be obtained for $\nabla_x G$, $\nabla_{xx} G$, $\nabla_{ux} G$. Therefore $G \in C^2$, and by the implicit function theorem [8, Theorem 4.2], for any $\bar{x} \in \mathcal{X}$ there exists an open neighborhood on which π_{f} is continuously differentiable. Ergo π_{f} is locally Lipschitz on $\mathcal{X}$. □

Lemma 2. *Consider the optimization problem*

$$v(x) = \underset{v \in \mathbb{R}^m}{\operatorname{argmin}} \|v - v_d(x)\|^2 \ \text{ s. t. } \ \|v\| \le r(x)$$

where $v_d : \mathcal{X} \to \mathbb{R}^m$, $r : \mathcal{X} \to \mathbb{R}_{\ge 0}$ are Lipschitz continuous with constants $L_{v_d}, L_r \ge 0$ respectively. Then $v : \mathcal{X} \to \mathbb{R}^m$ is also Lipschitz continuous with constant $L_{v_d} + L_r$.

Proof. For convenience, define $n(x) = \frac{v_d(x)}{\|v_d(x)\|}$, with the convention that if $\|v_d(x)\| = 0, n(x) = 0$. Notice v can be expressed as

$$v(x) = \mathrm{Proj}_{r(x)}(v_d(x)) = \min(r(x), \|v_d(x)\|) n(x) \quad (28)$$

where $\mathrm{Proj}_r(u)$ denotes the projection of $u \in \mathbb{R}^n$ onto the closed ball of radius $r \ge 0$ centered at the origin.

We aim to prove that there exists a $L \ge 0$ such that

$$\|v(x) - v(y)\| \le L \|x - y\|$$

for all $x, y \in \mathcal{X}$. Consider three cases:

Case 1 $[r(x) = r(y) = 0]$:
Since $v(x) = v(y) = 0$, the claim holds with $L = 0$.

Case 2 $[r(x) = 0, r(y) > 0]$:
Since $v(x) = 0$, and $v(y)$ is as in (28),

$$\begin{aligned} \|v(x) - v(y)\| &= \|\min(r(y), \|v_d(y)\|)\ n(y)\| \\ &= \min(r(y), \|v_d(y)\|)\ \|n(y)\| \\ &\le \min(r(y), \|v_d(y)\|) \\ &\le r(y) = |r(x) - r(y)| \end{aligned}$$

Hence the claim is true with $L = L_r$.

Case 3 $[r(x), r(y) > 0]$: Consider two subcases.

Case 3a $[\|v_d(x)\| \le r(x), \|v_d(y)\| \le r(y)]$:
Here, $v(x) = v_d(x), v(y) = v_d(y)$, and hence

$$\|v(x) - v(y)\| = \|v_d(x) - v_d(y)\| \le L_{v_d} \|x - y\|$$

thus, the claim holds with $L = L_{v_d}$.

Case 3b $[\|v_d(y)\| > r(y)]$: Let $p = \mathrm{Proj}_{r(x)}(v_d(y)) = r(x)n(y)$. Then,

$$\begin{aligned} &\|v(x) - v(y)\| \le \|v(x) - p\| + \|p - v(y)\| \\ &= \left\|\mathrm{Proj}_{r(x)} v_d(x) - \mathrm{Proj}_{r(x)} v_d(y)\right\| + \|r(x)n(y) - r(y)n(y)\|. \end{aligned}$$

The first term on the right is bounded by $\|v_d(x) - v_d(y)\|$ since the projection onto a closed convex set has Lipschitz constant 1. The second term is bounded by $|r(x) - r(y)|\,\|n(y)\|$. Since $\|n(y)\| \le 1$,

$$\begin{aligned} \|v(x) - v(y)\| &\le \|v_d(x) - v_d(y)\| + |r(x) - r(y)|\,\|n(y)\| \\ &\le L_{v_d} \|x - y\| + L_r \|x - y\|. \end{aligned}$$

Hence the claim is true with $L = L_{v_d} + L_r$. □